Semiclassical measures and the Schrödinger flow on Riemannian manifolds

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Abstract

In this article we study limits of Wigner distributions (the so-called semiclassical measures) corresponding to sequences of solutions to the semiclassical Schrödinger equation at times scales α_h tending to infinity as the semiclassical parameter h tends to zero (when $\alpha_h = 1/h$ this is equivalent to consider solutions to the non-semiclassical Schrödinger equation). Some general results are presented, among which a weak version of Egorov's theorem that holds in this setting. A complete characterization is given for the Euclidean space and Zoll manifolds (that is, manifolds with periodic geodesic flow) via averaging formulae relating the semiclassical measures corresponding to the evolution to those of the initial states. The case of the flat torus is also addressed; it is shown that non-classical behavior may occur when energy concentrates on resonant frequencies. Moreover, we present an example showing that the semiclassical measures associated to a sequence of states no longer determines those of their evolutions. Finally, some results concerning the equation with a potential are presented.

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1 Introduction

The *quantum-classical correspondence principle* roughly states that quantum systems behave according to classical mechanics in the high-frequency limit. A particular case that has attracted special attention corresponds to taking as the underlying classical system the geodesic flow on a complete Riemannian manifold (M, g). Its quantum counterpart is the Schrödinger flow, *i.e.* the unitary group $e^{ith\Delta/2}$ generated by the Laplace-Beltrami operator Δ on $L^2(M)$. In order to relate its high-frequency properties to the geodesic flow, one tries to determine the limiting behavior as $h \to 0$ of the position densities $|\psi_h(t, \cdot)|^2$ associated to solutions to the Schrödinger equation:

$$ih\partial_t \psi_h(t, x) + \frac{h^2}{2}\Delta\psi_h(t, x) = 0 \qquad (t, x) \in \mathbf{R} \times M, \tag{1}$$

issued from a sequence of highly oscillating initial data $\psi_h|_{t=0} = u_h$, whose characteristic lengths of oscillations are of order h. One expects that in this limit the dynamics of $|\psi_h(t, \cdot)|^2$ are somehow related to the geodesic flow.

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Usually, it is preferable to consider instead of $|\psi_h(t,\cdot)|^2$ the so-called *Wigner distribution* of ψ_h defined on the cotangent bundle T^*M. Given a solution $\psi(t,\cdot) = e^{ith\Delta/2}u \in L^2(M)$, its Wigner distribution $W_u^h(t,\cdot)$ acts on smooth, compactly supported test functions $a \in C_c^\infty(T^*M)$ as:

$$\left\langle W_u^h(t,\cdot), a\right\rangle := (\mathrm{op}_h(a)\, e^{ith\Delta/2}u | e^{ith\Delta/2}u). \tag{2}$$

Above, $(\cdot|\cdot)$ denotes the inner product in $L^2(M)$ and $\mathrm{op}_h(a)$ stands for the semiclassical pseudodifferential operator of symbol a obtained by Weyl's quantization rule (see Section 3 and the references therein for precise definitions and further properties of these objects). The Wigner distribution behaves, in some sense, as a joint position and momentum density: it is real, although not necessarily positive, and its marginals are precisely the position and momentum densities of ψ (a detailed presentation may be found, for instance, in the book [16]). Therefore, the limit of $|\psi_h(t,\cdot)|^2$ may be recovered from that of $W_u^h(t,\cdot)$ simply by projecting on M.

There are different regimes in which the correspondence principle can be made precise in the form of a rigorous result.

The semiclassical limit. Given a sequence of initial data (u_h) bounded in $L^2(M)$, consider the corresponding Wigner distributions $W_{u_h}^h$ given by (2). It is by now well-known that the distributions $W_{u_h}^h(t,\cdot)$ converge as $h \to 0^+$ (after possibly extracting a subsequence) to a family of positive measures $\mu(t,\cdot)$, continuous in time, usually called *semiclassical* or *Wigner measures*. It turns out that this limits are transported along the geodesic flow ϕ_t on T^*M:[1]

$$\lim_{h\to 0^+} (\mathrm{op}_h(a)\, e^{ith\Delta/2}u_h | e^{ith\Delta/2}u_h) = \int_{T^*M} a(\phi_{-t}(x,\xi))\, d\mu(0,x,\xi). \tag{3}$$

Moreover, if (u_h) oscillates at some characteristic length-scale h (see hypothesis (5) in Section 2) then the position densities $|e^{ith\Delta/2}u_h|^2$ weakly converge towards the marginal $\int_{T_x^*M} \mu(t,x,d\xi)$. This is the precise sense in which we recover classical dynamics as $h \to 0^+$ in this particular setting.

This kind of result holds in any compact Riemannian manifold, regardless of its particular geometric properties. However, only *small times* (of order h) are considered in the limit (3). This prevents the dispersive nature of the Schrödinger flow to become effective. Since the proof of (3) relies essentially on Egorov's theorem, statement (3) still holds for times of order $T_{\mathrm{E}}^h := C\log(1/h)$;[2] that is, when rescaling the Wigner distribution as $W_{u_h}^h\left(T_{\mathrm{E}}^h t,\cdot\right)$.

Eigenfunction limits. Another approach, which gives results that are valid for any time scale, consists of assuming that M is compact and taking as initial data eigenfunctions of $-\Delta$. If (ψ_{λ_k}) is a sequence of normalized eigenfunctions, $-\Delta\psi_{\lambda_k} = \lambda_k\psi_{\lambda_k}$ with $\lambda_k \to \infty$, then the corresponding solutions to the Schrödinger equation (1) are $e^{ith\Delta/2}\psi_{\lambda_k} = e^{-ith\lambda_k/2}\psi_{\lambda_k}$. The associated Wigner distributions act on $a \in C_c^\infty(T^*M)$ as:

$$(\mathrm{op}_h(a)\,\psi_{\lambda_k}|\psi_{\lambda_k}). \tag{4}$$

After setting $h = h_k = 1/\sqrt{\lambda_k}$ and taking limits in (4), a semiclassical measure is obtained (in this context, sometimes also called a *quantum limit*). Note that, since the Wigner distributions are time-independent, the limits of (4) are *uniform in time*. Moreover, quantum limits are invariant under the geodesic flow and are supported in the unit cosphere bundle S^*M. The main issue in

[1]This is a classical result that has been revisited and extended by many authors. A rigorous proof of this (and further results in that direction) in the recent mathematical literature may be found, for instance, in [18, 25, 7] (se also [30]).

[2]This is the *Ehrenfest time*, the time up to which, in a general system, a wavepacket remains localized (see for instance, [3, 6, 11, 21, 22, 33] for rigorous results in this direction).

this setting is that of identifying the set of all possible invariant measures on S^*M that can be realized as a quantum limit.

This problem is, in general, very hard and depends heavily on the specific geometry of the manifold under consideration. For instance, when the geodesic flow is ergodic the celebrated Schnirelman theorem asserts that for a sequence of eigenvalues of density one, (4) converges to the Liouville measure on S^*M. Therefore, most sequences of eigenfunctions become equidistributed on M (see the original article of Schnirelman [32] and [35, 10, 23, 20, 31, 1, 2], among many others, for various extensions and improvements). In the case of completely integrable geodesic flow the situation is quite different. For instance, when (M, g) is the sphere $\mathbf{S}^d$ equipped with the canonical metric Jakobson an Zelditch proved in [24] that every invariant measure on $S^*\mathbf{S}^d$ may be realized as a quantum limit for some sequence of normalized eigenfunctions.[3] For a more comprehensive account of the results quoted so far the reader may consult, for instance, [14, 30, 36].

Intermediate time scales. In this article we are interested in an intermediate regime. We shall analyze the structure of semiclassical measures arising as limits of Wigner distributions of solutions to a class of Schrödinger equations at time scales $t_h = \alpha_h$ tending to infinity as $h \to 0^+$; witch can be in principle much greater than the Ehrenfest time. One should expect that the dispersive effects associated to the Schrödinger flow would have to be taken into account.[4]

It turns out that the highly oscillating nature of the propagator $e^{it\Delta/2}$ prevents in general that the rescaled Wigner distributions $W^h_{u_h}(\alpha_h t, \cdot)$ converge for all $t \in \mathbf{R}$. Therefore, we shall study the relations between time-averages of (2) and the semiclassical measures of their corresponding sequences of initial data. The existence of these limits is established in Theorem 1; Theorem 2 shows that they are invariant by the geodesic flow (as was the case for eigenfunction limits) and that a weak from of Egorov's theorem holds for time scales $\alpha_h = o\left(h^{-2}\right)$.

Then, in order to get a more detailed description of these limits, we examine some examples of manifolds with completely integrable geodesic flow: Zoll manifolds, Euclidean space and the flat torus. Under some assumptions on the initial data we prove that limits of time averages of (2) are expressed as averages under the geodesic flow of the semiclassical measure of the initial states (see Theorem 4 and Propositions 6, 10, which are proved in Section 5). This is again a manifestation of the correspondence principle.

However, it turns out that such a behavior may fail in general, even in the completely integrable case. In Proposition 11, which is proved in Section 6, we present an example of initial data for which the semiclassical measure of the corresponding evolved states does not obey the averaging rule mentioned above. Instead of this, they evolve following a law related to the Schrödinger flow, thus exhibiting a genuinely quantum behavior. Moreover, we show that there is no longer a formula relating it to the semiclassical measure μ_0 of the sequence of initial states. It is possible to construct sequences having the same μ_0 but such that the limits of the time averages of (2) differ.

Finally, in Section 7 we discuss how our results extend to more general Schrödinger equations.

Notation and conventions. In what follows, (M, g) will always denote a connected, complete, d-dimensional smooth Riemannian manifold possessing a semiclassical functional calculus (property (F) in Section 3).

T^*M and S^*M stand for the cotangent and unit cosphere bundles on M respectively. Given a diffeomorphism $\Phi : M \to N$ between smooth manifolds, we will denote by $\tilde{\Phi} : T^*M \to T^*N$ the canonical diffeomorphisms induced by Φ.

[3] An extension of this result to general Compact Rank-One Symmetric Spaces can be found in [26].

[4] Recent results, perhaps of a different flavor, on semiclassical dynamics beyond the Ehrenfest time may be found, for instance, in [15, 33].

The Riemannian norm of a point $(x,\xi) \in T^*M$ is denoted by $\|\xi\|_x$.

The geodesic flow on T^*M is the Hamiltonian flow induced by the Riemannian energy $\frac{1}{2}\|\xi\|_x^2$. It will be denoted by ϕ_t.

The Riemannian measure in M will be denoted by dm. We shall write for short $L^2(M) := L^2(M, dm)$; the scalar product of two functions $u, v \in L^2(M)$ will be written as $(u|v)$.

The Riemannian gradient will be denoted by ∇; the Laplacian is denoted by $\Delta := \operatorname{div}(\nabla\cdot)$. It is a self-adjoint operator on $L^2(M)$.

$\mathcal{M}(T^*M)$ (resp. $\mathcal{M}_+(T^*M)$) denotes the space of Radon measures (resp. the cone of positive Radon measures) on T^*M. The space $\mathcal{M}(T^*M)$ may be identified, by Riesz's theorem, to the dual of the space of continuous compactly supported functions $C_c(T^*M)$.

A sequence of measures (μ_n) in $\mathcal{M}(T^*M)$ converges vaguely to some Radon measure μ as $n \to \infty$ if and only if $\lim_{n\to\infty} \int_{T^*M} a d\mu_n = \int_{T^*M} a d\mu$ for every $n \to \infty$ and $a \in C_c(T^*M)$.

A measure $\mu \in \mathcal{M}(T^*M)$ is invariant by a flow ϕ_t on T^*M if for any measurable set $X \subset T^*M$ one has $\mu(X) = \mu(\phi_s(X))$ for every $s \in \mathbf{R}$. This can be equivalently stated as $\int_{T^*M} a d\mu = \int_{T^*M} a \circ \phi_s d\mu$ for every $s \in \mathbf{R}$ and $a \in C_c(T^*M)$.

The space of compactly supported smooth functions on T^*M will be written as $C_c^\infty(T^*M)$; its dual, the space of distributions on T^*M, will be denoted by $\mathcal{D}'(T^*M)$. The duality bracket in $\mathcal{D}'(T^*M) \times C_c^\infty(T^*M)$ will be denoted by $\langle \cdot, \cdot \rangle$. Weak-$*$ convergence in $\mathcal{D}'(T^*M)$ will be simply referred to as weak convergence.

Given a set $A \subset \mathbf{R}$, its characteristic function will be denoted by $\mathbf{1}_A$.

2 Statement of the results

Our first results describe some properties of the limits of Wigner distributions at times $t = \alpha_h \to \infty$ corresponding to solutions to (1) on a general Riemannian manifold. In Section 7 we comment on extensions of these results to more general Schrödinger equations.

We shall make some hypotheses on the initial states. As it is also the case when dealing with the semiclassical limit, we shall assume that the admissible sequences of initial data (u_h) satisfy the h-oscillation property:

$$\limsup_{h\to 0^+} \left\| \mathbf{1}_{(-\infty,R)}\left(h^2\Delta\right)\phi u_h \right\|_{L^2(M)} \to 0, \qquad \text{as } R \to -\infty, \text{ for every } \phi \in C_c^\infty(M). \tag{5}$$

When the spectrum of Δ is discrete, this roughly means that the energy of u_h is concentrated on Fourier modes corresponding to eigenvalues of size at most R/h^2.

Moreover, we shall assume that their Wigner distributions converge to some semiclassical measure $\mu_0 \in \mathcal{M}_+(T^*M)$:

$$\lim_{h\to 0^+} (\operatorname{op}_h(a) u_h | u_h) = \int_{T^*M} a(x,\xi)\,\mu_0(dx, d\xi), \tag{6}$$

for every $a \in C_c^\infty(T^*M)$. This is always achieved by some subsequence (provided that (u_h) is bounded in $L^2(M)$). See Proposition 12 and, in general, Section 3 for notation and background concerning pseudodifferential operators and Wigner distributions.

Unless otherwise stated, we shall denote by (α_h) a sequence of positive reals tending to infinity as $h \to 0^+$.

Theorem 1 *Let (u_h) be a bounded sequence in $L^2(M)$ satisfying hypotheses (5) and (6). Then there exist a subsequence and a finite measure $\mu \in L^\infty(\mathbf{R}_t;\mathcal{M}_+(T^*M))$ such that the following statements hold.*

i) For every $\varphi \in L^1(\mathbf{R})$ and every $a \in C_c^\infty(T^*M)$,

$$\lim_{h\to 0^+} \int_{\mathbf{R}} \varphi(t) \left(\mathrm{op}_h(a) e^{i\alpha_h ht\Delta/2} u_h | e^{i\alpha_h ht\Delta/2} u_h\right) dt = \int_{\mathbf{R}\times T^*M} \varphi(t)\, a(x,\xi)\, \mu(t, dx, d\xi)\, dt. \quad (7)$$

ii) For every $\varphi \in L^1(\mathbf{R})$ and $a \in C_c(M)$ the evolved position densities satisfy:

$$\lim_{h\to 0+} \int_{\mathbf{R}\times M} \varphi(t)\, a(x) \left|e^{i\alpha_h ht\Delta/2} u_h(x)\right|^2 dm dt = \int_{\mathbf{R}\times T^*M} \varphi(t)\, a(x)\, \mu(t, dx, d\xi)\, dt. \quad (8)$$

In general, the convergence in (7) does not hold pointwise. Several examples of such a behavior will be presented in our next results.

Theorem 2 *Let μ and μ_0 be obtained as a limit (7) and (6), respectively. Then the following hold.*

i) For almost every $t \in \mathbf{R}$, the measure $\mu(t, \cdot)$ is invariant under the geodesic flow ϕ_s, i.e.

$$\mu(t, \phi_s(\Omega)) = \mu(t, \Omega), \qquad \text{for every } s \in \mathbf{R} \text{ and } \Omega \subset T^*M \text{ measurable.} \quad (9)$$

ii) If $a \in C_c^\infty(T^*M)$ is invariant under the classical flow and $\alpha_h = o\left(1/h^2\right)$ then the following holds pointwise, for every $t \in \mathbf{R}$*:*

$$\lim_{h\to 0^+} \left(\mathrm{op}_h(a) e^{i\alpha_h ht\Delta/2} u_h | e^{i\alpha_h ht\Delta/2} u_h\right) = \int_{T^*M} a(x,\xi)\, \mu_0(dx, d\xi). \quad (10)$$

Remark 3 *The restriction $\alpha_h = o\left(1/h^2\right)$ in part ii) of the theorem can be removed in some cases, as the Euclidean space $\mathbf{R}^d$ or the flat torus $\mathbf{T}^d$. Its presence is related to the commutation properties with Δ of $\mathrm{op}_h(a)$ when a is invariant. Further details are given in Remark 16, in Section 4.*

Part i) is a consequence of the time averaging over large time intervals. Part ii) establishes that we can still keep track of the pointwise behavior of the Wigner distributions at large time scales, provided we test them against an invariant classical symbol. This can be interpreted as a weak form of Egorov's theorem for long times. An analogue of Theorem 1 and of part i) of Theorem 2 in terms of microlocal defect measures (see for instance [8, 17] for background) can be found in [13].

In order to obtain a more detailed description and, in particular, to derive formulas that allow to compute μ in terms of the semiclassical measure of the initial data μ_0, we must restrict the geometry of the manifolds under consideration. We shall consider examples of manifolds with completely integrable geodesic flow

We first consider the case of *Zoll manifolds* (that is, manifolds all whose geodesics are closed). We refer to the book [4] for a comprehensive study of this geometric hypothesis. Such manifolds are compact, and the restriction of the geodesic flow ϕ_t to the unit cosphere bundle S^*M is periodic. Given a function $a \in C_c(T^*M)$ we write $\langle a \rangle$ to denote the average of a along the geodesic flow:

$$\langle a \rangle (x,\xi) := \lim_{T\to\infty} \frac{1}{T} \int_0^T a(\phi_s(x,\xi))\, ds. \quad (11)$$

Since, by the homogeneity of the flow, every trajectory is periodic, the limit above always exists. Moreover, $\langle a \rangle$ is bounded and measurable.

Theorem 4 *Suppose (M,g) is a manifold all of whose geodesics are closed and $\alpha_h = o\left(1/h^2\right)$. Let μ_0 be the semiclassical measure given by (6) for some sequence of initial data satisfying (5). If $\mu_0(\{\xi=0\})=0$ then any limit μ given by (7) is characterized by:*

$$\int_{T^*M} a(x,\xi)\,\mu(t,dx,d\xi) = \int_{T^*M} \langle a\rangle(x,\xi)\,\mu_0(dx,d\xi), \qquad \text{for a.e. } t\in\mathbf{R}. \tag{12}$$

In particular, if (u_h) is such that $\mu_0(x,\xi) = \delta_{x_0}(x)\,\delta_{\xi_0}(\xi)$ for some $(x_0,\xi_0)\in T^*M\setminus\{0\}$ –(u_h) is then called a wave-packet, see Proposition 14– then $\mu=\delta_\gamma$ is the Dirac delta on the geodesic γ issued from (x_0,ξ_0).[5] From this, using a diagonal argument, it is clear that if (M,g) is a Zoll manifold then every measure on $T^*M\setminus\{0\}$ that is invariant under the geodesic flow can be realized as a limit (7) for some sequence of initial data. This can be seen as a time dependent version of the result of Jakobson and Zelditch [24] for eigenfunctions of the Laplacian on the sphere we quoted in the introduction. Actually, Theorem 4 can also be applied to obtain results on quantum limits; in particular, it can be used to extend the result in [24] to a general Compact Rank-One Symmetric Space (see [26]).

Remark 5 *As it will be clear from the proof, Theorem 4 holds locally in the following sense. If the geometric hypothesis on (M,g) is replaced by the weaker:* there exist an open set $X\subset T^*M$, invariant under the geodesic flow, such that $\phi_s|_X$ is periodic on each of the cospheres $\|\xi\|_x$ =constant, *then (12) holds for every $a\in C_c^\infty(X)$.*

The proof of the theorem follows from a general result which relates the smoothness properties of the averages $\langle a\rangle$ to the time-pointwise behavior of Wigner distributions (*cf.* Lemma 17 in Section 4, which is of independent interest).

The consequence of the corresponding result on Euclidean space is trivial.

Proposition 6 *Suppose $(M,g)=\left(\mathbf{R}^d,\mathrm{can}\right)$ and (u_h) is a sequence that satisfies (5) and (6). If its semiclassical measure μ_0 satisfies $\mu_0(\{\xi=0\})=0$ then any measure μ given by (7) vanishes identically. In other words,*

$$\lim_{h\to0^+}\int_{\mathbf{R}}\varphi(t)\,(\mathrm{op}_h(a)\,e^{i\alpha_h ht\Delta/2}u_h|e^{i\alpha_h ht\Delta/2}u_h)dt = 0, \qquad \text{for every } a\in C_c^\infty\left(T^*\mathbf{R}^d\right).$$

Note that Proposition 6 can also be deduced from the $H^{1/2}$-regularizing effect of the Schrödinger equation (see for instance [12]).

Remark 7 *The condition $\mu_0(\{\xi=0\})=0$ roughly means that the sequence (u_h) cannot develop oscillations at frequencies lower than h^{-1}. It holds when*

$$\limsup_{h\to0^+}\left\|\mathbf{1}_{(\delta,0]}\left(h^2\Delta\right)\phi u_h\right\|_{L^2(M)}\to 0, \qquad \text{as } \delta\to0^-, \text{ for every } \phi\in C_c^\infty(M). \tag{13}$$

Remark 8 *On any Riemannian manifold, one easily checks that the limit (7) corresponding to the constant sequence $u_h := f\in L^2(M)$ is given, for every $t\in\mathbf{R}$, by:*

$$\mu(t,x,\xi) = |e^{it\Delta/2}f(x)|^2dx\delta_0(\xi).$$

Thus, the conclusions of Theorem 4 and Proposition 6 may not hold when $\mu_0(\{\xi=0\})\neq 0$.

[5] That is, δ_γ is the unique invariant probability measure on T^*M which is concentrated on γ. This is sometimes also called the orbit measure corresponding to γ.

Remark 9 *Analogues of these results hold for Schrödinger equations with a potential, see Theorem 19 and Remark 21 in Section 7.*

Our last set of results deal with the flat torus $\left(\mathbf{T}^d, \mathrm{can}\right)$. We shall identify $\mathbf{T}^d$ with the quotient $\mathbf{R}^d/\left(2\pi\mathbf{Z}\right)^d$ and $T^*\mathbf{T}^d$ to $\mathbf{T}^d \times \mathbf{R}^d$. Consider the set of *resonant frequencies*:

$$\Omega := \left\{\xi \in \mathbf{R}^d \ : \ k \cdot \xi = 0 \text{ for some } k \in \mathbf{Z}^d \setminus \{0\}\right\}.$$

We again get an averaging type result, provided our sequence of initial data does not concentrate on Ω.

Proposition 10 *Suppose μ and μ_0 are given respectively by (7) and (6) for some sequence (u_h) bounded in $L^2\left(\mathbf{T}^d\right)$ satisfying (5). If $\mu_0\left(\mathbf{T}^d \times \Omega\right) = 0$ then, for a.e. $t \in \mathbf{R}$ and every $a \in C_c^\infty\left(T^*\mathbf{T}^d\right)$,*

$$\int_{\mathbf{T}^d} a\left(x,\xi\right)\mu\left(t,dx,d\xi\right) = \int_{\mathbf{T}^d} \langle a\rangle\left(x,\xi\right)\mu_0\left(dx,d\xi\right).$$

As we mentioned in the introduction, the results obtained so far reflect that the correspondence principle holds. It turns out that this is no longer the case if μ_0 charges $\mathbf{T}^d \times \Omega$.

Proposition 11 *Let $\xi_0 \in \mathbf{Z}^d$, $\varrho \in L^2\left(\mathbf{T}^d\right)$ and $\alpha_h = 1/h$. Then there exist sequences (u_h) and (v_h) whose semiclassical measure is:*

$$\mu_0\left(x,\xi\right) = \left|\varrho\left(x\right)\right|^2 dx\delta_{\xi_0}\left(\xi\right), \tag{14}$$

but such that the limiting semiclassical measure (in the sense of (7)) for $\left(e^{it\Delta/2}u_h\right)$ is:

$$\mu_{(u_h)}\left(t,x,\xi\right) = \left\langle |e^{it\Delta/2}\varrho|^2\right\rangle(x)\,dx\delta_{\xi^0}\left(\xi\right), \tag{15}$$

(with $\langle\cdot\rangle$ defined by (11)), whereas that of $\left(e^{it\Delta/2}v_h\right)$ is given by:

$$\mu_{(v_h)}\left(t,x,\xi\right) = \frac{1}{\left(2\pi\right)^d}\left(\int_{\mathbf{T}^d}\left|\varrho\left(y\right)\right|^2 dy\right) dx\delta_{\xi^0}\left(\xi\right). \tag{16}$$

We can extract two consequences of this result. First, that the measures $\mu\left(t,\cdot\right)$ may have an explicit dependence on t, which is related to the Schrödinger flow and does not depend exclusively on the classical dynamics. Second, that no formula exists in general relating μ_0 and μ in the case that μ_0 charges the resonant set $\mathbf{T}^d \times \Omega$. In fact, μ depends on the way in which concentration of the sequence of initial data takes place on $\mathbf{T}^d \times \Omega$. A more detailed study requires the introduction of two-microlocal objects describing such a concentration and will be presented in [27].

3 Semiclassical measures

In this section we shall recall the necessary notions of semiclassical pseudodifferential calculus and semiclassical measures that will be needed in the sequel. We shall closely follow the presentation in [20]. Unless otherwise specified, we implicitly refer to [20] for complete proofs of the results presented in this section.

The classical *Weyl quantization rule* on $\mathbf{R}^d$ associates to any function $a \in C_c^\infty\left(\mathbf{R}^d \times \mathbf{R}^d\right)$ and any $h > 0$ an operator $\mathrm{op}_h\left(a\right)$ acting on $u \in C_c^\infty\left(\mathbf{R}^d\right)$ as:

$$\mathrm{op}_h\left(a\right)u\left(x\right) := \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} a\left(\frac{x+y}{2}, h\xi\right)u\left(y\right)e^{i(x-y)\cdot\xi}dy\frac{d\xi}{\left(2\pi\right)^d}.$$

It turns out that, under suitable growth conditions on a, the operators $\mathrm{op}_h(a)$ are uniformly bounded in $L^2(\mathbf{R}^d)$ when h ranges any compact set of the positive reals.

In order to extend this rule to functions $a \in C^\infty(T^*M)$ we shall do the following. Let $\kappa : U \subset \mathbf{R}^d \to V \subset M$ be a coordinate patch; assume that a is supported on $T^*M|_V$. Then define, for every $h > 0$, an operator $\mathrm{op}_h(a)$ by the formula:

$$(\mathrm{op}_h(a)\, u) \circ \kappa := \theta\, \mathrm{op}_h(a \circ \tilde{\kappa})\, (\theta u \circ \kappa),$$

where $a \circ \tilde{\kappa}$ is the expression of a in the coordinates κ and $\theta \in C_c^\infty(V)$ is identically equal to one on the projection of $\operatorname{supp} a$ on M. To deal with the general case it suffices to decompose the function a in compactly supported components using a partition of unity.

In what follows, we shall assume that $a \in C_c^\infty(T^*M)$. The operators $\mathrm{op}_h(a)$ are called *semiclassical pseudodifferential operators of symbol* a. The following facts are well known.

(A) The operators $\mathrm{op}_h(a)$ are bounded in $L^2(M)$ with norm:

$$\|\mathrm{op}_h(a)\|_{\mathcal{L}(L^2(M))} \leq C\, \|a\|_{C^{d+1}(T^*M)}, \tag{17}$$

the constant $C > 0$ being uniform in $h \in (0, 1]$.

(B) The family $\mathrm{op}_h(a)$ of operators is not completely determined by the function a – in fact, the result may depend on the partition of unity, the coordinate patches and the cut-off functions θ chosen. However the $L^2(M)$-operator norm of the difference of any two families defined from a by means of the above procedure tends to zero as $h \to 0^+$.

(C) The Laplacian of (M, g) may be expressed in terms of semiclassical pseudodifferential operators. One easily checks that

$$-h^2 \Delta = \mathrm{op}_h(p) + ih\, \mathrm{op}_h(r) + h^2\, \mathrm{op}_h(m),$$

where $m \in C^\infty(M)$ is a function of x alone depending only on the derivatives up to order two of the Riemannian metric g. In a coordinate chart κ, the functions p, r are given by:

$$(p \circ \tilde{\kappa})(x, \xi) := \sum_{i,j=1}^{d} g^{ij}(x)\, \xi^i \xi^j, \quad (r \circ \tilde{\kappa})(x, \xi) := \frac{1}{\rho(x)} \sum_{i,j=1}^{d} g^{ij}(x)\, \partial_{x_i} \rho(x)\, \xi_j,$$

where $\rho := \sqrt{\det g}$. Therefore, p coincides with the squared Riemannian norm $\|\xi\|_x^2$ and

$$r = \frac{1}{2} \{p, \log \rho\}, \tag{18}$$

where $\{\cdot, \cdot\}$ stands for the Poisson bracket induced by the canonical symplectic structure on T^*M.

The Weyl quantization rule enjoys a powerful symbolic calculus (see [14, 28] for a thorough description). Some particular cases are the following.

(D) *Commutators.* For every $a \in C_c^\infty(T^*M)$ and $h > 0$ there exists an operator $s_h \in \mathcal{L}(L^2(M))$ such that:

$$\left[\mathrm{op}_h(a), -h^2 \Delta\right] = \frac{h}{i}\, \mathrm{op}_h(\{a, p\}) + h^2\, \mathrm{op}_h(\{a, r\}) + s_h, \tag{19}$$

and $\|s_h\|_{\mathcal{L}(L^2(M))} \leq Ch^3$.

(E) *Adjoints.* If $a \in C_c^\infty(T^*M)$ is real then $\mathrm{op}_h(a)$ is self-adjoint in $L^2(M)$.

Finally, we shall assume that our manifold (M,g) possesses a semiclassical functional calculus. More precisely, that the following holds:

(F) *Functional calculus.* For every $\sigma \in C_c^\infty(\mathbf{R})$ the following holds:

$$\sigma\left(-h^2\Delta\right) = \mathrm{op}_h(\sigma \circ p) + z_h, \tag{20}$$

with $\|z_h\|_{\mathcal{L}(L^2(M))} \leq Ch$.

This is known to hold when M is compact (see [9]), and has been proved for Euclidean spaces in [30] and recently for manifolds with ends in [5].

Given a function $u \in L^2(M)$ we define its *Wigner distribution* $w_u^h \in \mathcal{D}'(T^*M)$ acting on test functions $a \in C_c^\infty(T^*M)$ as:

$$\left\langle w_u^h, a\right\rangle := \left(\mathrm{op}_h(a)\, u | u\right).$$

Property (E) of the Weyl quantization ensures that w_u^h is real. Moreover, the following result holds.

Proposition 12 *Let (u_h) be a bounded sequence in $L^2(M)$. Then for some subsequence (which we do not relabel) the Wigner distributions $w_{u_h}^h$ converge to a finite, positive Radon measure $\mu \in \mathcal{M}_+(T^*M)$:*

$$\lim_{h\to 0^+} \left\langle w_{u_h}^h, a\right\rangle = \int_{T^*M} a(x,\xi)\, \mu_0(dx, d\xi), \qquad \textit{for all } a \in C_c^\infty(T^*M). \tag{21}$$

Note that property (B) of $\mathrm{op}_h(\cdot)$ ensures that the limit μ_0 does not depend on the partitions of unity, coordinate charts, and cut-off functions used to define $\mathrm{op}_h(a)$.

Whenever (21) holds, we say that μ_0 is the *semiclassical measure* of the sequence (u_h). If in addition the sequence satisfies the h-oscillation property (5) then $|u_h|^2\, dm$ tends to the projection on M of μ_0 as $h \to 0^+$.

Proposition 13 *Let μ_0 be the semiclassical measure of an h-oscillating sequence (u_h). Suppose that*

$$|u_h|^2\, dm \rightharpoonup \nu \qquad \textit{vaguely in } \mathcal{M}_+(M) \textit{ as } h \to 0^+.$$

Then

$$\int_{T_x^*M} \mu_0(x, d\xi) = \nu(x).$$

Proof. The proof of this result combines that of Proposition 1.6 in [20] with the functional calculus formula (20). Working in coordinates $\kappa : U \subset \mathbf{R}^d \to V \subset M$ and following exactly the reasoning in [20], Proposition 1.6, we deduce that the conclusion holds provided

$$\limsup_{h\to 0^+} \int_{|\xi|>R/h} \left|\widehat{\theta u_h \circ \kappa}(\xi)\right|^2 d\xi \to 0, \text{ as } R \to \infty,$$

for any $\theta \in C_c^\infty(V)$. Using the functional calculus formula (20) we deduce that this condition is satisfied whenever (5) holds. ■

We conclude this review of semiclassical measures examining a specific computation of the semiclassical measure of a wave-packet. Let $(x_0, \xi_0) \in T^*M$ and (U, κ) a coordinate system

centered at x_0 (*i.e.* $0 \in U$ and $\kappa(0) = x_0$). Let $\varrho \in C_c^\infty(\mathbf{R}^d)$ be supported in U and identically equal to one near the origin and let $\varphi \in C^\infty(M)$ be a function such that for $x \in U$,

$$\varphi(\kappa(x)) = \kappa^*(\xi_0) \cdot x + i|x|^2,$$

where $\kappa^*(\xi_0)$ stands for the pull-back by κ of the covector $\xi_0 \in T^*_{x_0}M$. Define $\rho_h \in C_c^\infty(\kappa(U))$ as $\rho_h(\kappa(x)) := \varrho(x/h^{1/2})$ and $v_h \in L^2(M)$ as

$$v_h(x) := Ch^{-d/4}\rho_h(x)\, e^{i\varphi(x)/h},$$

where $C > 0$ is chosen to have $\|v_h\|_{L^2(M)} = 1$.

The sequence (v_h) is called a *wave-packet (or a coherent state) centered at* (x_0, ξ_0). A simple computation shows the following.

Proposition 14 *The sequence (v_h) is h-oscillatory and has a semiclassical measure $\mu_0 = \delta_{(x_0,\xi_0)}$.*

Using an orthogonality property of semiclassical measures (see [18], Proposition 3.3) and the preceding result one sees that every linear combination of delta measures in T^*M can be realized as the semiclassical measure of some sequence in $L^2(M)$. Since these combinations of point masses are dense in $\mathcal{M}_+(T^*M)$, by the Krein-Milman theorem, we conclude that *every finite, positive Radon measure on T^*M can be realized as the semiclassical measure for some sequence in $L^2(M)$.*

4 Proof of Theorems 1 and 2

Proof of Theorem 1. Let $\psi_h(t,x) := e^{i\alpha_h ht\Delta/2}u_h(x)$ and consider the corresponding sequence of time-space Wigner distributions $W_h \in \mathcal{D}'(T^*(\mathbf{R} \times M))$ defined by

$$\langle W_h, b\rangle := (\mathrm{op}_h(b_h)\,\psi_h|\psi_h)_{L^2(\mathbf{R}\times M)},$$

where, for $b \in C_c^\infty(T^*(\mathbf{R}\times M))$ we have written $b_h(t,x,\tau,\xi) := b(t,x,\tau/\alpha_h,\xi)$. It is easy to check that sequence (W_h) is bounded in $\mathcal{D}'(T^*(\mathbf{R}\times M))$, therefore it is possible to extract a subsequence (which we shall not relabel) such that

$$\lim_{h\to 0^+}\langle W_h, b\rangle = \int_{T^*(\mathbf{R}\times M)} b(t,x,\tau,\xi)\, d\tilde{\mu}(t,x,\tau,\xi).$$

It turns out (see [7, 20]) that the limit $\tilde{\mu}$ is a positive Radon measure on $T^*(\mathbf{R}\times M)$. Let $\varphi, \chi \in C_c^\infty(\mathbf{R})$, with $0 \le \chi \le 1$ and $\chi|_{(-1,1)} \equiv 1$. For every $a \in C_c^\infty(T^*M)$ we can write:

$$\int_{\mathbf{R}} \varphi(t)\,(\mathrm{op}_h(a)\,\psi_h|\psi_h)\,dt = \left(\mathrm{op}_h\left(b_h^R\right)\psi_h|\psi_h\right)_{L^2(\mathbf{R}\times M)} + r(R,h), \tag{22}$$

where $b_h^R(t,x,\tau,\xi) := \varphi(t)\,\chi(\tau/\alpha_h R)\,a(x,\xi)$ and the remainder r is defined as follows. Set $\sigma_R(\tau) := \sqrt{1-\chi(\tau/R)}$; standard arguments of semiclassical pseudodifferential calculus give

$$r(R,h) = \int_{\mathbf{R}} \varphi(t)\,(\mathrm{op}_h(a)\,\sigma_R\left(\frac{h}{\alpha_h}D_t\right)\psi_h|\sigma_R\left(\frac{h}{\alpha_h}D_t\right)\psi_h)_{L^2(\mathbf{R}\times M)}dt + \mathcal{O}\left(h^2\right).$$

We have used the notation $\sigma_R\left(\frac{h}{\alpha_h}D_t\right)$ to denote the operator $\mathrm{op}_{\frac{h}{\alpha_h}}(\sigma_R)$ acting on functions defined on $\mathbf{R}_t$. Clearly, $\sigma_R\left(\frac{h}{\alpha_h}D_t\right)\psi_h = \sigma_R\left(h^2\Delta/2\right)\psi_h$; therefore,

$$|r(R,h)| \le C_{a,\varphi}\left\|\sigma_R\left(h^2\Delta/2\right)\psi_h\right\|^2_{L^2(M)} + \mathcal{O}\left(h^2\right),$$

and (5) ensures that $\limsup_{h\to 0^+} r(R,h)$ tends to 0 as $R\to\infty$. Taking limits in (22), first in $h\to 0^+$ then $R\to\infty$, we conclude:

$$\lim_{h\to 0^+}\int_{\mathbf{R}}\varphi(t)\left(\mathrm{op}_h(a)\psi_h|\psi_h\right)dt=\int_{T^*(\mathbf{R}\times M)}\varphi(t)\,a(x,\xi)\,d\tilde{\mu}(t,x,\tau,\xi)\,. \tag{23}$$

Note that, because of the bound:

$$\sup_{t\in\mathbf{R}}\left|\left(\mathrm{op}_h(a)\,e^{i\alpha_h ht\Delta/2}u_h|e^{i\alpha_h ht\Delta/2}u_h\right)\right|\le C\,\|a\|_{C^{d+1}(T^*M)}\,\|u_h\|^2_{L^2(M)}\,,$$

convergence in (23) actually takes place for any $\varphi\in L^1(\mathbf{R})$ and the limit is in $L^\infty(\mathbf{R};\mathcal{M}_+(T^*M))$. Therefore, the measure $\mu(t,x,\xi):=\int_{\mathbf{R}}\tilde{\mu}(t,x,d\tau,\xi)$ fulfills the requirements of i).

We now prove ii). First remark that we cannot directly derive (8) from part i), since test functions depending only on x are not compactly supported in T^*M. We start noticing that $\left|e^{i\alpha_h ht\Delta/2}u_h\right|^2$ is bounded in $L^\infty\left(\mathbf{R};L^1(M)\right)$; this ensures existence of the limit in (8), eventually for a subsequence. In order to identify the limit it is better to work locally in a coordinate patch $\kappa: U\subset\mathbf{R}^d\to V\subset M$. From the functional calculus identity (20) and the h-oscillation hypothesis (5) one deduces that, for any $\theta\in C_c^\infty(V)$ and $\varphi\in L^1(\mathbf{R})$, the sequence $(\theta\psi_h\circ\kappa)$ enjoys the (euclidean) h-oscillation property:

$$\limsup_{h\to 0^+}\int_{\mathbf{R}}\int_{|\xi|>R/h}\varphi(t)\,|\widehat{\theta\psi_h\circ\kappa}(t,\xi)|^2d\xi dt\to 0,\qquad\text{as } R\to\infty. \tag{24}$$

From this it is easy to conclude (8) following the lines of the proof of [20], Proposition 1.6. In fact, for $a\in C_c(V)$,

$$\int_M a(x)\left|e^{i\alpha_h ht\Delta/2}u_h(x)\right|^2dm=\int_{\mathbf{R}}\varphi(t)\left(\mathrm{op}_h(a_R)\,e^{i\alpha_h ht\Delta/2}u_h|e^{i\alpha_h ht\Delta/2}u_h\right)dt+r(R,h)$$

where $a_R(x,\xi):=a(x)\chi(\xi/R)$ for some $\chi\in C_c^\infty\left(\mathbf{R}^d\right)$ with $\chi(0)=1$, $0\le\chi\le 1$, and $\limsup_{h\to 0} r(R,h)\to 0$ as $R\to\infty$ because of (24). ■

Proof of Theorem 2. A direct computation shows:

$$\frac{d}{dt}\left(\mathrm{op}_h(a)\,e^{i\alpha_h ht\Delta/2}u_h|e^{i\alpha_h ht\Delta/2}u_h\right)=\frac{i\alpha_h h}{2}\left(\left[\mathrm{op}_h(a),\Delta\right]e^{i\alpha_h ht\Delta/2}u_h|e^{i\alpha_h ht\Delta/2}u_h\right). \tag{25}$$

Given $\varphi\in C_c^\infty(\mathbf{R})$, identities (25) and (19) ensure:

$$\frac{1}{\alpha_h}\int_{\mathbf{R}}\varphi'(t)\left(\mathrm{op}_h(a)\,e^{i\alpha_h ht\Delta/2}u_h|e^{i\alpha_h ht\Delta/2}u_h\right)dt= \tag{26}$$

$$=\frac{1}{2}\int_{\mathbf{R}}\varphi(t)\left(\mathrm{op}_h(\{a,p\})\,e^{i\alpha_h ht\Delta/2}u_h|e^{i\alpha_h ht\Delta/2}u_h\right)dt+\int_{\mathbf{R}}\varphi(t)\,d_h(t)\,dt, \tag{27}$$

where $d_h(t)\le Ch\|u_h\|^2_{L^2(M)}$. Taking limits, we conclude that for every $a\in C_c^\infty(T^*M)$ and almost every $t\in\mathbf{R}$:

$$\int_{T^*M}\{a,p\}(x,\xi)\,\mu(t,dx,d\xi)\,dt=0, \tag{28}$$

and therefore prove i).

Now we turn to the proof of ii). If the symbol $a \in C_c^\infty(T^*M)$ is ϕ_s-invariant then $\{a,p\}=0$; in this case (19) and (25) give:

$$\begin{aligned}&(\mathrm{op}_h(a)\,e^{i\alpha_h ht\Delta/2}u_h|e^{i\alpha_h ht\Delta/2}u_h)-(\mathrm{op}_h(a)\,u_h|u_h)\\&=-\alpha_h h\frac{i}{2}\int_0^t(\mathrm{op}_h(\{a,r\})\,e^{i\alpha_h hs\Delta/2}u_h|e^{i\alpha_h hs\Delta/2}u_h)ds-\int_0^t f_h(s)\,ds,\end{aligned}$$

with $f_h := 2\alpha_h h^{-2}\left(s_h e^{i\alpha_h ht\Delta/2}u_h|e^{i\alpha_h ht\Delta/2}u_h\right)\leq C\left\|u_h\right\|^2_{L^2(M)}o(1)$. Taking imaginary and real parts, we infer, respectively:

$$\lim_{h\to 0^+}\alpha_h h\int_0^t(\mathrm{op}_h(\{a,r\})\,e^{i\alpha_h hs\Delta/2}u_h|e^{i\alpha_h hs\Delta/2}u_h)ds=0, \tag{29}$$

and, for every $t\in\mathbf{R}$,

$$\lim_{h\to 0+}(\mathrm{op}_h(a)\,e^{i\alpha_h ht\Delta/2}u_h|e^{i\alpha_h ht\Delta/2}u_h)=\lim_{h\to 0+}(\mathrm{op}_h(a)\,u_h|u_h),$$

which is precisely (10). ■

Remark 15 *Equation (29) does not give any new information about the semiclassical measures $\mu(t,\cdot)$. Applying Jacobi's identity, formula (18), and using the invariance of a we obtain:*

$$\{a,r\}=\frac{1}{2}\{a,\{p,\log\rho\}\}=-\frac{1}{2}\{p,\{\log\rho,a\}\}.$$

Therefore (29) may be restated as $\int_{T^*M}\{p,\{\log\rho,a\}\}\,d\mu=0$, which was already deduced from the invariance property (28), since equations (26) and (27) imply:

$$\alpha_h h\int_{\mathbf{R}}\varphi(t)\,(\mathrm{op}_h(\{b,p\})\,e^{i\alpha_h ht\Delta/2}u_h|e^{i\alpha_h ht\Delta/2}u_h)dt=o(1),$$

for every symbol $b\in C_c^\infty(T^*M)$.

Remark 16 *Note that the restriction $\alpha_h=o\left(h^{-2}\right)$ may be removed as soon as we have $[\mathrm{op}_h(a),\Delta]=0$ for every invariant $a\in C_c^\infty(T^*M)$. This is the case when M is either the Euclidean space or the flat torus, for instance.*

5 Averaging formulae

Now we turn to the proof of Theorem 4. Our first remark concerns the case in which the average $\langle a\rangle$ of a symbol is smooth.

Lemma 17 *Let μ and μ_0 be as in Theorem 1 and $\alpha_h=o\left(1/h^2\right)$. Suppose that $a\in C_c^\infty(T^*M)$ is such that $\langle a\rangle$ is infinitely differentiable in T^*M. Then, for almost every $t\in\mathbf{R}$,*

$$\int_{T^*M}a(x,\xi)\,\mu(t,dx,d\xi)=\int_{T^*M}\langle a\rangle(x,\xi)\,\mu_0(dx,d\xi).$$

Proof. From statement ii) in Theorem 2 we infer, noticing that $\langle a\rangle$ is necessarily ϕ_s-invariant, that, for a.e. $t\in\mathbf{R}$,

$$\int_{T^*M}\langle a\rangle(x,\xi)\,\mu(t,dx,d\xi)=\int_{T^*M}\langle a\rangle(x,\xi)\,\mu_0(dx,d\xi).$$

Now, taking into account that $\mu(t,\cdot)$ is ϕ_s-invariant for a.e. $t \in \mathbf{R}$ and using the dominated convergence theorem, we deduce:

$$\begin{aligned}\int_{T^*M} \langle a\rangle (x,\xi)\, \mu(t,dx,d\xi) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{T^*M} a(\phi_s(x,\xi))\, \mu(t,dx,d\xi)\, ds \\ &= \int_{T^*M} a(x,\xi)\, \mu(t,dx,d\xi),\end{aligned}$$

as claimed. ■

Assume that all the geodesics of (M,g) are closed. This implies (see [4]) that there exists $L>0$ such that, for every $(x,\xi) \in T^*M$ with $\|\xi\|_x = 1$, the geodesic

$$\mathbf{R} \ni s \longmapsto \phi_s(x,\xi) \in T^*M$$

is L-periodic. As a consequence of homogeneity, the geodesic corresponding to a general $(x,\xi) \in T^*M \setminus \{0\}$ is $L/\|\xi\|_x$-periodic.

Proof of Theorem 4. Let $a \in C_c^\infty(T^*M)$ vanish in a neighborhood of $\{\xi = 0\}$. Due to the periodicity of the geodesic flow, the average of a equals:

$$\langle a\rangle (x,\xi) := \frac{\|\xi\|_x}{L}\int_0^{L/\|\xi\|_x} a(\phi_s(x,\xi))\, ds,$$

for every $(x,\xi) \in T^*M$. It follows that $\langle a\rangle$ is a smooth function; using Lemma 17 we conclude that identity (12) holds for a. This implies that $\mu(t, T^*M\setminus\{0\}) = \mu_0(T^*M\setminus\{0\})$ for a.e. $t\in\mathbf{R}$. Since M is compact, (8) implies that, again for a.e. $t\in\mathbf{R}$, the total masses of $\mu(t,\cdot)$ and μ_0 are equal. Finally, as $\mu_0(\{\xi=0\}) = 0$ we must necessarily have $\mu(t,\{\xi=0\}) = 0$ and formula (12) follows for arbitrary $a \in C_c^\infty(T^*M)$. ■

Proof of Proposition 6. The proof is immediate: for almost every $t\in\mathbf{R}$ the measures $\mu(t,\cdot)$ are invariant by translations $(x,\xi)\mapsto(x+s\xi,\xi)$ (by Theorem 2, i)) and do not charge the set $\{\xi=0\}$, as the projection of $\mu(t,\cdot)$ on ξ coincides with that of μ_0 (this can be checked directly, or seen as a consequence of Theorem 2, ii)). This and the fact that $\mu(t,\cdot)$ is finite for a.e. t forces $\mu = 0$. ■

Proof of Proposition 10. Let $a \in C_c^\infty\left(T^*\mathbf{T}^d\right)$, and consider its average $\langle a\rangle$ along the geodesic flow. The hypothesis $\mu_0\left(\mathbf{T}^d\times\Omega\right) = 0$ ensures that, for μ_0-almost every $\xi\in\mathbf{R}^d$ we have

$$\langle a\rangle (x,\xi) = \overline{a}(\xi) := \frac{1}{(2\pi)^d}\int_{\mathbf{T}^d} a(y,\xi)\, dy,$$

as only dense geodesics are involved in the average. We cannot apply Lemma 17 in this setting, since $\langle a\rangle$ is not smooth. However, by Theorem 2, ii) (note that there is no restriction on α_h, by Remark 16), we have that $\int_{\mathbf{T}^d}\mu(t,dx,\cdot) = \int_{\mathbf{T}^d}\mu_0(dx,\cdot)$ and therefore, for *a.e.* $t\in\mathbf{R}$,

$$\int_{T^*\mathbf{T}^d} \langle a\rangle (x,\xi)\, \mu(t,dx,d\xi) = \int_{T^*\mathbf{T}^d} \overline{a}(\xi)\, \mu_0(dx,d\xi).$$

We apply the dominated convergence theorem and use the invariance of $\mu(t,\cdot)$ under the geodesic flow to conclude

$$\int_{T^*\mathbf{T}^d} a(x,\xi)\, \mu(t,dx,d\xi) = \int_{T^*\mathbf{T}^d} \langle a\rangle (x,\xi)\, \mu(t,dx,d\xi),$$

for *a.e.* $t\in\mathbf{R}$, and the proof follows. ■

6 Concentration on resonant frequencies

In this section we prove Proposition 11. From now on, we shall identify functions defined on $\mathbf{T}^d$ to the $2\pi\mathbf{Z}^d$-periodic functions defined on $\mathbf{R}^d$. If so, the Euclidean Wigner distribution of

$$u(x) = \sum_{k\in\mathbf{Z}^d} \widehat{u}(k)\frac{e^{ik\cdot x}}{(2\pi)^{d/2}} \in L^2\left(\mathbf{T}^d\right)$$

is given by:

$$l_u^h(x,\xi) := \sum_{k,j\in\mathbf{Z}^d} \widehat{u}(k)\overline{\widehat{u}(j)}\frac{e^{i(k-j)\cdot x}}{(2\pi)^d}\delta_{\frac{h}{2}(k+j)}(\xi).$$

It is easy to check that l_u^h differs from the Wigner distribution w_u^h defined in Section 3 by an $\mathcal{O}(h)$ term. Therefore, their limits coincide and give the usual semiclassical measures. Clearly, $L_u^h(t,\cdot) := l^h_{e^{i\alpha_h ht\Delta/2}u_h}$ satisfies,

$$\int_{\mathbf{R}}\varphi(t)\left\langle L_u^h(t,\cdot),a\right\rangle dt = \frac{1}{(2\pi)^{d/2}}\sum_{k,j\in\mathbf{Z}^d}\widehat{\varphi}\left(\frac{|k|^2-|j|^2}{2}\right)\widehat{u}(k)\overline{\widehat{u}(j)}a_{j-k}\left(\frac{hk+hj}{2}\right),$$

for every $a\in C_c^\infty\left(T^*\mathbf{T}^d\right)$ of the form $a(x,\xi) = (2\pi)^{-d/2}\sum_{k\in\mathbf{Z}^d}a_k(\xi)e^{ik\cdot x}$.

We now define the sequences (u_h) and (v_h). Let $\theta_0\in\mathbf{R}^d\backslash\Omega$; let (k_n) be a sequence in $\mathbf{Q}^d$ such that $\lim_{n\to\infty}k_n=\theta_0$. Suppose that $k_n = \left(p_n^1/q_n^1,...,p_n^d/q_n^d\right)$ with p_n^j and q_n^j relatively prime; let q_n denote the least common multiple of q_n^1,...,q_n^d and write $\lambda_1 := q_1$, and $\lambda_n = q_n\lambda_{n-1}$ for $n>1$. Clearly, only a finite number of the q_n may be equal to one, therefore $\lim_{n\to\infty}\lambda_n=\infty$. Finally, set $h_n := 1/(\lambda_n)^2$.

Now write,

$$S^1(x) := \xi_0\cdot x, \qquad S_n^2(x) := \xi_0\cdot x+\sqrt{h_n}k_n\cdot x,$$

and

$$u_{h_n}(x) := \varrho(x)e^{iS^1(x)/h_n}, \qquad v_{h_n}(x) := \varrho(x)e^{iS_n^2(x)/h_n}.$$

Since $\lambda_n^2\xi_0,\lambda_nk_n\in\mathbf{Z}^d$, the Fourier coefficients of u_{h_n} and v_{h_n} are obtained from those of ϱ as:

$$\widehat{u_{h_n}}(k) = \widehat{\varrho}\left(k-\lambda_n^2\xi_0\right), \qquad \widehat{v_{h_n}}(k) = \widehat{\varrho}\left(k-\lambda_n^2\xi_0-\lambda_nk_n\right).$$

The proof of the fact that the limits of $\left(l^{h_n}_{u_{h_n}}\right)$, $\left(l^{h_n}_{v_{h_n}}\right)$ coincide with the measure given by (14) is simple and may be reconstructed following the same lines as that for the evolution case. We therefore concentrate on the latter.

Let us now compute the limit of $\left(L^{h_n}_{u_{h_n}}\right)$; clearly, it suffices to consider the limit against test functions $a_l(x,\xi) := b(\xi)e^{-il\cdot x}$ with $b\in C_c^\infty\left(\mathbf{R}^d\right)$ and $\varphi\in L^1(\mathbf{R})$ with $\widehat{\varphi}\in C_c(\mathbf{R})$. We can write:

$$\begin{aligned}\int_{\mathbf{R}}\varphi(t)\left\langle L^{h_n}_{u_{h_n}}(t,\cdot),a_l\right\rangle dt &= \sum_{k-j=l} b\left(\frac{h_nk+h_nj}{2}\right)\widehat{\varphi}\left(\frac{|k|^2-|j|^2}{2}\right)\widehat{u_{h_n}}(k)\overline{\widehat{u_{h_n}}(j)}\\ &= \sum_{k-j=l} b\left(h_n\frac{k+j}{2}+\xi_0\right)\widehat{\varphi}\left(l\cdot\left(\frac{k+j}{2}+\lambda_n^2\xi_0\right)\right)\widehat{\varrho}(k)\overline{\widehat{\varrho}(j)}.\end{aligned}$$

If $l\cdot\xi_0\neq 0$ then the expression above vanishes as $n\to\infty$. To see this, suppose that $\operatorname{supp}\widehat{\varphi}\subset(-R,R)$; clearly:

$$\left|\int_{\mathbf{R}}\varphi(t)\left\langle L^{h_n}_{u_{h_n}}(t,\cdot),a_l\right\rangle dt\right|\leq\|b\|_{L^\infty(\mathbf{R}^d)}\left|\sum_{j\in\mathbf{Z}^d}\widehat{\varphi}\left(l\cdot\left(j+\frac{l}{2}+\lambda_n^2\xi_0\right)\right)\widehat{\varrho}(j+l)\overline{\widehat{\varrho}(j)}\right|. \tag{30}$$

The distance d_n between the hyperplane $l\cdot\left(\xi+l/2+\lambda_n^2\xi_0\right)=0$ and the origin tends to infinity as $n\to\infty$. Therefore, for n large enough we can estimate (30) by

$$\|b\|_{L^\infty(\mathbf{R}^d)}\|\widehat{\varphi}\|_{L^\infty(\mathbf{R})}\sum_{|j|>d_n-2R}\left|\widehat{\varrho}(j+l)\overline{\widehat{\varrho}(j)}\right|,$$

which tends to zero as $n\to\infty$ since $\varrho\in L^2\left(\mathbf{T}^d\right)$.

When $l\cdot\xi_0=0$ we have:

$$\int_{\mathbf{R}}\varphi(t)\left\langle L^{h_n}_{u_{h_n}}(t,\cdot),a_l\right\rangle dt=\sum_{k-j=l}b\left(h_n\frac{k+j}{2}+\xi_0\right)\widehat{\varphi}\left(\frac{|k|^2-|j|^2}{2}\right)\widehat{\varrho}(k)\overline{\widehat{\varrho}(j)},$$

and letting $n\to\infty$ gives (recall that $\widehat{\varphi}$ is compactly supported):

$$\int_{\mathbf{R}\times\mathbf{T}^d}\varphi(t)\,a_l(\xi)\,\mu_{(u_h)}(t,dx,d\xi)=b(\xi_0)\sum_{k-j=l}\widehat{\varphi}\left(\frac{|k|^2-|j|^2}{2}\right)\widehat{\varrho}(k)\overline{\widehat{\varrho}(j)}.$$

In conclusion, for a general $a\in C_c^\infty\left(T^*\mathbf{T}^d\right)$ of the form $a(x,\xi):=\sum_{l\in\mathbb{Z}^d}b_l(\xi)\,e^{-il\cdot x}$ one has:

$$\begin{aligned}\int_{\mathbf{R}\times\mathbf{T}^d}\varphi(t)\,a(x,\xi)\,\mu_{(u_h)}(t,dx,d\xi)&=\sum_{l\cdot\xi_0=0}\sum_{k-j=l}b_l(\xi_0)\,\widehat{\varphi}\left(\frac{|k|^2-|j|^2}{2}\right)\widehat{\varrho}(k)\overline{\widehat{\varrho}(j)}\\&=\int_{\mathbf{T}^d}\langle a\rangle(x,\xi_0)\left|e^{it\Delta/2}\rho(x)\right|^2dx,\end{aligned}$$

($\langle a\rangle$ being defined by (11)) and therefore (15) holds for (u_{h_n}).

Now we turn to the corresponding computation for $\left(L^{h_n}_{v_{h_n}}\right)$. Reasoning as before, we have:

$$\left|\int_{\mathbf{R}}\varphi(t)\left\langle L^{h_n}_{v_{h_n}}(t,\cdot),a_l\right\rangle dt\right|\leq\|b\|_{L^\infty(\mathbf{R}^d)}\left|\sum_{j\in\mathbf{Z}^d}\widehat{\varphi}\left(l\cdot\left(j+\frac{l}{2}+\lambda_n^2\xi_0+\lambda_nk_n\right)\right)\widehat{\varrho}(j+l)\overline{\widehat{\varrho}(j)}\right|.$$

Now, if $l\neq 0$ it is easy to check that distance between the hyperplane $l\cdot\left(\xi+l/2+\lambda_n^2\xi_0+\lambda_nk_n\right)=0$ and the origin always tends to infinity, since $\lim_{n\to\infty}l\cdot k_n=l\cdot\theta_0\neq 0$ and therefore, $\lim_{n\to\infty}\left|l\cdot\left(l/2+\lambda_n^2\xi_0+\lambda_nk_n\right)\right|=\infty$. The same argument we used for (u_{h_n}) now gives us:

$$\lim_{n\to\infty}\int_{\mathbf{R}}\varphi(t)\left\langle L^{h_n}_{v_{h_n}}(t,\cdot),a_l\right\rangle dt=0.$$

When $l=0$ we have

$$\int_{\mathbf{R}}\varphi(t)\left\langle L^{h_n}_{u_{h_n}}(t,\cdot),a_l\right\rangle dt=\widehat{\varphi}(0)\sum_{j\in\mathbf{Z}^d}b\left(h_nj+\sqrt{h_n}k_n+\xi_0\right)|\widehat{\varrho}(j)|^2,$$

which converges precisely to $(2\pi)^{-d}\,\widehat{\varphi}(0)\,b(\xi_0)\,\|\varrho\|^2_{L^2(\mathbf{T}^d)}$. This shows that (16) holds.

7 Schrödinger equations with a potential

Some of the results presented here have an analogue for the more general Schrödinger equation:

$$ih\partial_t \psi_h(t,x) + \frac{h^2}{2}\Delta\psi_h(t,x) - V(x)\psi_h(t,x) = 0 \qquad (t,x) \in \mathbf{R} \times M. \tag{31}$$

provided we assume that the potential $V \in C^2(M)$ satisfies:

$$\text{the Hamiltonian flow } \phi_t^H \text{ on } T^*M \text{ associated to } H(x,\xi) := \frac{1}{2}\|\xi\|_x^2 + V(x) \text{ is complete;} \tag{32}$$

$$\text{the operator } \mathcal{H}_h := \frac{h^2}{2}\Delta - V \text{ is essentially self-adjoint in } L^2(M). \tag{33}$$

Note that both conditions are met when, for instance, $V \geq -C$ for some $C > 0$. See [34, 29] and the references therein for a thorough discussion on this issue.

The analogues of Theorem 1 and 2 hold for the evolved Wigner distributions in this framework:

$$\left\langle W_{u_h}^h(t,\cdot), a\right\rangle := (\mathrm{op}_h(a) e^{it/h\mathcal{H}_h} u_h | e^{it/h\mathcal{H}_h} u_h).$$

Theorem 18 *Let (u_h) be a sequence bounded in $L^2(M)$ satisfying conditions (5) (with $h^2\Delta$ replaced by $\mathcal{H}_h$) and (6). Let μ_0 be its semiclassical measure. Then, at least for some subsequence, the following hold.*

i) There exists a measure $\mu \in L^\infty(\mathbf{R}; \mathcal{M}_+(T^*M))$ such that

$$\lim_{h\to 0^+} \int_{\mathbf{R}} \varphi(t) \left\langle W_{u_h}^h(\alpha_h t, \cdot), a\right\rangle dt = \int_{\mathbf{R}} \varphi(t) \int_{T^*M} a(x,\xi)\,\mu(t,dx,d\xi)\,dt,$$

for every $\varphi \in L^1(\mathbf{R})$ and $a \in C_c^\infty(T^*M)$.

ii) For a.e.t $\in \mathbf{R}$, the measure $\mu(t,\cdot)$ is invariant under the Hamiltonian flow ϕ_s^H.

iii) Given $a \in C_c^\infty(T^*M)$ invariant under the classical flow ϕ_t^H and $\alpha_h = o(1/h^2)$, the following holds:

$$\lim_{h\to 0^+} \left\langle W_{u_h}^h(\alpha_h t, \cdot), a\right\rangle = \int_{T^*M} a(x,\xi)\,\mu_0(dx,d\xi), \qquad \text{for every } t \in \mathbf{R}.$$

This is a consequence of the fact that the presence of the potential V does not introduce terms of order h^2 in the expansion for the commutator:

$$[\mathrm{op}_h(a), \mathcal{H}_h] = \frac{h}{i}\mathrm{op}_h(\{a,H\}) + h^2\,\mathrm{op}_h(\{a,r\}) + \mathcal{O}(h^3).$$

It is easy to prove using Lemma 17 the following analogue of Theorem 4 in this setting.

Theorem 19 *Suppose that the Hamiltonian flow ϕ_t^H is periodic and $\alpha_h = o(1/h^2)$. Let μ_0 be the semiclassical measure of some sequence (u_h) in $L^2(M)$ satisfying (5) and (6). Then, for any subsequence for which (7) holds we have the averaging formula:*

$$\lim_{h\to 0^+} \int_{\mathbf{R}} \varphi(t) \left\langle W_{u_h}^h(\alpha_h t, \cdot), a\right\rangle dt = \left(\int_{\mathbf{R}} \varphi(t)\,dt\right) \int_{T^*M} \langle a\rangle(x,\xi)\,\mu_0(dx,d\xi), \tag{34}$$

for every $a \in C_c^\infty(T^*M)$ and $\varphi \in L^1(\mathbf{R})$, the average $\langle a\rangle$ being taking with respect to ϕ_t^H.

Remark 20 *If ϕ_t^H is just periodic in $X := H^{-1}(E_1, E_2)$ for some $E_1 < E_2$, then formula (34) holds for functions $a \in C_c^\infty(X)$.*

Remark 21 *The conclusions of Theorem 4 and Proposition 6 also hold for the solutions to the adimensional equation:*

$$i\partial_t v_h + \frac{1}{2}\Delta v_h - V v_h = 0,$$

as they can be written as solutions the semiclassical equation (31) with potential $h^2 V$ evaluated at time t/h. Therefore, as an immediate consequence of the proof, the conclusions of Theorem 2 hold with ϕ_s being the geodesic flow of (M, g).

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